PMM U.S.S.R.,Vol.51,No.5,pp.675-681,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00


# ASYMPTOTIC SOLUTION OF ELASTICITY THEORY PROBLEMS ON CRACKS EXTENDED ALONG A SPACE CURVE*

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A class of spatial problems of elasticity theory for cracks extended along a smooth space curve is solved by asymptotic methods. The two first terms of the asymptotic form of the displacement jumps and the stress intensity coefficients are constructed and their dependence on the crack geometry is investigated. The problem of an annular crack on a cylindrical surface is considered as an example, and results of its asymptotic solution are presented for different kinds of loads, including taking account of superposition of the edges.

Analogous, more simple problems on cracks extended along a plane curve are considered in /1, 2/.

**1. General equations for an extended crack.** We consider a homogeneous, isotropic, infinite elastic space containing a crack extended along a smooth space curve $\mathbf{R} = \mathbf{R}(l)$, $l \in [-L, L]$ (the latter can be closed: $\mathbf{R}(-L) = \mathbf{R}(L)$). We introduce tangent, normal, and binormal directions $\mathbf{t}(l)$, $\mathbf{v}(l)$, $\mathbf{b}(l)$ to the curve $\mathbf{R}$ that satisfy the Frenet relationships (the prime denotes the derivative with respect to $l$)

$$\mathbf{R}' = \mathbf{t}, \quad \mathbf{t}' = k\mathbf{v}, \quad \mathbf{v}' = -k\mathbf{t} + \tau\mathbf{b}, \quad \mathbf{b}' = -\tau\mathbf{v} \tag{1.1}$$

where $k$, $\tau$ are the curvature and torsion of the curve $\mathbf{R}$.

We define allowable crack shapes by the relationships

$$|l| \leqslant L, \ |m_1| \leqslant \rho(l), \ m_2 = 0 \tag{1.2}$$

where the coordinates $(m_1, m_2, l)$ are given by the equality

$$\mathbf{x}(m_1, m_2, l) = \mathbf{R} + \varepsilon [m_1 \boldsymbol{\alpha}^1(l) + m_2 \boldsymbol{\alpha}^2(l)] \tag{1.3}$$

$\varepsilon$ is a positive small parameter; the orthonormalized vector triplet $\boldsymbol{\alpha}^i(l)$ $(i = 1, 2, 3)$ is obtained from the triplet $\mathbf{v}(l), \mathbf{b}(l), \mathbf{t}(l)$ by rotation around $\mathbf{t}(l)$ by an angle $\varphi(l)$:

$$\boldsymbol{\alpha}^1 = \mathbf{v}\cos\varphi + \mathbf{b}\sin\varphi, \quad \boldsymbol{\alpha}^2 = -\mathbf{v}\sin\varphi + \mathbf{b}\cos\varphi, \quad \boldsymbol{\alpha}^3 = \mathbf{t}$$

the functions $\rho(l)$ and $\varphi(l)$ are sufficiently smooth, where $\rho / L = O(1)$. For unclosed cracks we also require that $\rho(L) = \rho(-L) = 0$ (for closed cracks simply $\rho(L) = \rho(-L)$).

The crack given by relationships (1.2) is a narrow strip of rectilinear section stretched along the middle line of $\mathbf{R}(l)$ for small $\varepsilon$, where the function $\rho(l)$ and $\varphi(l)$, respectively, describe the change in crack width and its orientation relative to the accompanying trihedron $\mathbf{t}, \mathbf{v}, \mathbf{b}$. In particular, rotation, flexure, and torsion of the crack surface are allowed. The directions $\boldsymbol{\alpha}^3$ and $\boldsymbol{\alpha}^1$ here yield the longitudinal and transverse directions on the crack surface, while $\boldsymbol{\alpha}^2$ is the normal direction to the crack surface at its middle line (for $m_1 = 0$). The vector triplet $\boldsymbol{\alpha}^i$ is rotated according to the following law as it moves along $\mathbf{R}$ (as is easily obtained from (1.1):

$$\boldsymbol{\alpha}^{i\prime} = \mathbf{A} \times \boldsymbol{\alpha}^i, \quad \mathbf{A} = A_i \boldsymbol{\alpha}^i, \quad A_1 = k\sin\varphi, \quad A_2 = k\cos\varphi,$$
$$A_3 = \tau + \varphi'$$

(here and henceforth, the summation is assumed over repeated subscripts). The components $A_i(l)$ $(i = 1, 2, 3)$ have the meaning, respectively, of rates of flexure, rotation "in its plane" and torsion of the crack surface.

The class of cracks introduced above includes a number of cracks on spatial surfaces, for instance along the arc of a circle of a cylinder or cone, cracks along a spiral line, etc., as well as all extended cracks of planar planform considered earlier /1, 2/.

We assume that there is no load at infinity while the forces

$$\mathbf{p}^+(\mathbf{x}) = -\mathbf{p}^-(\mathbf{x}) = \mathbf{p}(\mathbf{x}), \quad \mathbf{x} \in G_\varepsilon$$

*Prikl.Matem.Mekhan.,51,5,858-865,1987

are applied to the crack surfaces, where the plus and minus signs refer to the "upper" and "lower" surfaces bounding the domains $m_2 > 0$ and $m_2 < 0$, respectively. (As is well-known, the boundary conditions mentioned correspond to the problem of external elastic field perturbation by a crack in a space without cracks). Determination of the elastic fields reduces to seeking the displacement jump $\varkappa(\mathbf{x})$ on the crack surface

$$\varkappa(\mathbf{x}) = \mathbf{u}^+(\mathbf{x}) - \mathbf{u}^-(\mathbf{x}), \quad \mathbf{x} \in G_\varepsilon \tag{1.4}$$

We will find the asymptotic form of the displacement jump on the crack surface as $\varepsilon \to 0$. Let $\mathbf{n}(\mathbf{x})$ $(\mathbf{x} \in G_\varepsilon)$ be a vector normal to the surface $G_\varepsilon$

$$\begin{aligned} &\mathbf{n}(m_1, l) = [d\mathbf{x}/dl \times \boldsymbol{\alpha}^1(l)]/|d\mathbf{x}/dl \times \boldsymbol{\alpha}^1(l)| = \\ &\quad \boldsymbol{\alpha}^2(l) - \varepsilon m_1 A_3(l)\boldsymbol{\alpha}^3(l) + o(\varepsilon) \\ &\mathbf{p}(m_1, m_2, l) = -\sigma_{ij}(m_1, m_2, l)\, n_j(m_1, l)\,\mathbf{e}_i, \quad \mathbf{p}(m_1, 0, l) = \mathbf{p}(m_1, l) \end{aligned} \tag{1.5}$$

where $\mathbf{p}$ is a force vector.

Then (/3/, formulas (12) and (10)), it follows from the Somigliani formula that

$$\begin{aligned} &p_m(x) = \beta\{(1-\nu)\, n_s(\mathbf{x})\, [\varphi_{(mi)}(\mathbf{x})_{,is} + \varphi_{(is)}(\mathbf{x})_{,im}] + \\ &\quad 2\nu\, [n_s(\mathbf{x})\, \varphi_{ii}(\mathbf{x})_{,sm} + n_m(\mathbf{x})\, \varphi_{is}(\mathbf{x})_{,is}] - n_s(\mathbf{x})\, \psi_{it}(\mathbf{x})_{,istm}\} \\ &\beta = 4\pi\mu^{-1}(1-\nu), \quad \mathbf{x} \in G_\varepsilon \end{aligned} \tag{1.6}$$

where $\mu$, $\nu$ are the shear modulus and Poisson's ratio of the medium; symmetrization is over the subscripts in parentheses, and the harmonic and biharmonic potentials $\varphi_{ij}$ and $\psi_{ij}$ are determined by the formulas

$$\begin{aligned} &\varphi_{ij}(\mathbf{x}) = \iint_{G_\varepsilon} \varkappa_i(\mathbf{x}')\, n_j(\mathbf{x}')\, |\Delta\mathbf{x}|^{-1}\, d\mathbf{x}', \quad \psi_{ij}(\mathbf{x}) = \iint_{G_\varepsilon} \varkappa_i(\mathbf{x}')\, n_j(\mathbf{x}')\, |\Delta\mathbf{x}|\, d\mathbf{x}' \\ &\Delta\mathbf{x} = \mathbf{x}' - \mathbf{x}, \quad \mathbf{x} \in G_\varepsilon \end{aligned} \tag{1.7}$$

We find the asymptotic of the integrodifferential operator acting on $\varkappa(\mathbf{x})$ in (1.6) near the crack in the coordinates $m_1, m_2, l$ $(m_2 \neq 0)$, and then by letting $m_2 \to 0$ we obtain an asymptotic equation in $\varkappa(m_1, l, \varepsilon)$ from (1.6). The principal term of the asymptotic form of the operator should obviously correspond to the operators of the plane and antiplane problems. Consequently, if $p$ denotes the characteristic magnitude of the load $|\mathbf{p}(m_1, l)|$, the principal term of the expansion of $\varkappa$ should be of the order $\varepsilon\rho(l)\, p/\mu \sim \varepsilon L p/\mu$.

Let us determine in what order (as compared with the principal term of the asymptotic form) the mutual influence of the different parts of the crack is felt (which is at a distance $\sim L$). After substituting (1.7) into (1.6) and differentiating, the kernel of the integrals are of the order $\mu\,|\Delta\mathbf{x}|^{-3} \sim \mu L^{-3}$, and integration over the domain with the area $\sim\varepsilon\rho L \sim \varepsilon L^2$ yields a quantity of the order of $\varepsilon L^{-3} \times \varepsilon L^2 \times \varepsilon L p/\mu = \varepsilon^2 p$ which is two orders of magnitude less than the given load. Therefore, the mutual influence of parts of the crack should be felt only in the third term of the asymptotic form $\varkappa$ (in powers of $\varepsilon$). We limit ourselves in this paper to seeking the first two terms of the asymptotic form $\varkappa$ in the general case, which by virtue of the above exposition should be "local", i.e., depend in each section $l = l_0$ of the crack on just the geometry and load in the neighbourhood of this section.

Taking into account that

$$\psi_{it}(\mathbf{x})_{,it} = \varphi_{ii} - \psi^*(\mathbf{x}), \quad \psi^*(\mathbf{x}) = \iint_{G_\varepsilon} (\Delta\mathbf{x}, \varkappa(\mathbf{x}'))(\Delta\mathbf{x}, \mathbf{n}(\mathbf{x}'))\, |\Delta\mathbf{x}|^{-1} d\mathbf{x}'$$

relationship (1.6) can be written in the form

$$\begin{aligned} &p_m = \beta\{(1-\nu)\, n_s\, [\varphi_{(mt),ts} + \varphi_{(st),tm}] + 2\nu n_m \varphi_{st,st} - \\ &\quad (1-2\nu)\, n_s \varphi_{ii,sm} + n_s \psi^*_{,sm}\}, \quad \mathbf{x} \in G_\varepsilon \end{aligned} \tag{1.8}$$

As is seen from (1.8), it is necessary to find the asymptotic expansion of $\varphi_{ij}$, $\psi^*$ and the operator $\partial^2/\partial x_i \partial x_j$ in the coordinates $m_1, m_2, l$ as $\varepsilon \to 0$. It can be shown for the latter that

$$\begin{aligned} &\frac{\partial^2}{\partial x_i \partial x_j} = \varepsilon^{-2} \alpha_i^s \alpha_j^t \frac{\partial^2}{\partial m_s \partial m_t} + \varepsilon^{-1}\Big[(-1)^{s+1} m_{3-s} A_3 a_{ij}^t \frac{\partial^2}{\partial m_s \partial m_t} + \\ &\quad a_{ij}^t \frac{\partial^2}{\partial m_t \partial l} + (-1)^t (A_{3-t} \alpha_i^3 \alpha_j^3 - A_3 a_{ij}^t) \frac{\partial}{\partial m_t}\Big] + O(1), \quad a_{ij}^t = \alpha_i^t \alpha_j^3 + \alpha_j^t \alpha_i^3 \end{aligned} \tag{1.9}$$

where $s$ and $t$ run through the values 1 and 2.

The asymptotic form of the integrals $\varphi_{ij}$ and $\psi^*$ is calculated by using the composite asymptotic expansions of the integrands as in /1, 2/. This asymptotic form has a rather cumbersome form; we present just those of its terms that are needed to evaluate the first two terms of the asymptotic form $\varkappa$:

$$\begin{aligned}
&\varphi_{ij}(\mathbf{x})/\varepsilon = -\alpha_j^2 L_i(1) + \varepsilon[A_3\alpha_j^3 L_i(m_1') + {}^1/_2 A_3\alpha_j^2 L_i(g_s)] + \\
&\quad \varphi_{ij}^0(l) + \varepsilon\varphi_{ij}^1(m_1, m_2, l) + o(\varepsilon) \\
&\psi^*(\mathbf{x})/\varepsilon = 2\alpha_i^3 K_i(g_s g_2) + \varepsilon[\alpha_i^3 \partial L_i(g_2)/\partial l - A_3\alpha_i^3 L_i(g_1) - \\
&\quad 2A_3\alpha_i^3 K_i(m_1' g_2^2) + {}^1/_2 A_s\alpha_i^3 L_i(g_s) + (-1)^s\alpha_i^t K_i(g_s g_t g_2)] + \\
&\quad \psi_0^*(l) + \varepsilon\psi_1^*(m_1, m_2, l) + o(\varepsilon) \\
&g_1 = m_1 - m_1', \quad g_2 = m_2; \quad \varkappa_i = \varkappa_i(m_1', l), \quad \forall f(m_1, m_1', m_2) \\
&L_i(f) = \int \varkappa_i f \ln(g_1^2 + m_2^2)\, dm_1', \quad K_i(f) = \int \varkappa_i f (g_1^2 + m_2^2)^{-1}\, dm_1'
\end{aligned} \tag{1.10}$$

Integration here is performed from $-\rho(l)$ to $\rho(l)$, $\varphi_{ij}^0, \psi_0^*$ are independent of $m_1, m_2$, $\varphi_{ij}^1, \psi_1^*$ depend linearly on $m_1, m_2$ (these components vanish in the calculation of the first two terms of the asymptotic form on differentiation by means of (1.8), taking (1.9) into account); $s$ and $t$ run through the values 1 and 2.

Assuming that

$$\mathbf{p}(\mathbf{x}) = \mathbf{p}^0(m_1, l) + \varepsilon\mathbf{p}^1(m_1, l) + o(\varepsilon), \quad \mathbf{x} \in G_\varepsilon \tag{1.11}$$

and taking account of (1.8) and the form of the asymptotic forms (1.9) and (1.10), we seek the asymptotic form $\varkappa(m_1, l, \varepsilon)$ as $\varepsilon \to 0$ in the form

$$\varkappa(m_1, l, \varepsilon) = \varepsilon[\varkappa^0(m_1, l) + \varepsilon\varkappa^1(m_1, l) + o(\varepsilon)] \tag{1.12}$$

Substituting the asymptotic form (1.9)-(1.12) into (1.8), determining the limits of the integrals $K$ and $L$ as $m_2 \to 0$ (to do this the latter are expressed in terms of the convolution of generalized functions dependent on the parameter $m_2$) and equating terms of the same order in $\varepsilon$, we obtain integrodifferential equations for $\varkappa^0, \varkappa^1$. In writing them we use an orthonormalized $\varepsilon$-dependent triplet of vectors $\gamma^1(m_1, l) = \alpha^1(m_1, l)$, $\gamma^2(m_1, l) = \mathbf{n}(m_1, l)$, $\gamma^3(m_1, l) = \gamma^1 \times \gamma^3$ governing, respectively, the transverse, normal to the surface, and longitudinal directions at each point of the crack surface. (Note that because of twisting of the crack generally $\gamma^2(m_1, l) \neq \alpha^2(l)$ and $\gamma^3(m_1, l) \neq \alpha^3(l)$ for $m_1 \neq 0$.)

If the quantity $\varkappa_i^0(m_1, l), \varkappa_i^1(m_1, l)$ is introduced such that

$$\begin{aligned}
&\varkappa(m_1, l, \varepsilon) = \varepsilon[\varkappa_i^0(m_1, l) + \varepsilon\varkappa_i^1(m_1, l)]\gamma^i(m_i, l) + o(\varepsilon^2) \\
&\text{i.e. } \varkappa_i^0 = (\varkappa^0, \alpha^i) \text{ and } \varkappa_1^1 = (\varkappa^1, \alpha^1), \ \varkappa_j^1 = \\
&\quad (\varkappa^1, \alpha^j) - (-1)^j \varepsilon m_1 A_3(\varkappa^0, \alpha^{5-j}) \quad (j = 2, 3),
\end{aligned} \tag{1.13}$$

and in an analogous manner, the quantities $p_i^0(m_1, l)$ and $p_i^1(m_1, l)$, then the desired integrodifferential equations are written in the form

$$\begin{aligned}
&2(1 - \nu\delta_{i3}) P(\varkappa_i^0)_{,m_1 m_1} = \beta p_i^0 \\
&2(1 - \nu\delta_{i3}) P(\varkappa_i^1)_{,m_1 m_1} = \beta p_i^1 + H_{ij}(\varkappa_j^0)
\end{aligned} \tag{1.14}$$

where the operators $P$ and $H_{ij}$ are defined as follows:

$$\begin{aligned}
&P(\varphi) = \int_{-\rho(l)}^{\rho(l)} \varphi(m_1', l) \ln|m_1 - m_1'|\, dm_1' \\
&H_{ii} = (1 - \nu\delta_{i3}) A_2 P_{,m_1}, \ H_{12} = -H_{21} = (1 - 4\nu) A_1 P_{,m_1} \\
&H_{23} = -H_{32} = -2A_3 P_{,m_1}, \ H_{13} = -H_{31} = -2\nu P_{,m_1 l}
\end{aligned}$$

Relationships (1.14) obtained for $m_1 \leqslant \rho(l)$ are equations of the plane and antiplane problems for a rectilinear crack, as might have been expected, and can be solved successively in quadratures. In the special case of a crack of plane planform $(\varphi = \tau = A_1 = A_3 = 0)$ Eqs. (1.14) agree (to an accuracy of the order under consideration) with the equations obtained earlier /1, 2/. In conformity with the above, (1.14) possess the property of "being local".

It can also be confirmed (by using (1.14) and the symmetry or antisymmetry of the integrodifferential operators $P_{,m_1}$, $P_{,m_1 m_1}$, $P_{,m_1 l}$ in the sense of the scalar product in $L_2(G)$ that the form of the operators $H_{ij}$ (particularly the coefficients $(1 - \nu\delta_{i3}) A_2$ in the diagonal terms as well as the symmetry (antisymmetry) of the non-diagonal terms) ensures the validity of the Betti theorem in the first two terms of the asymptotic. In addition, this confirms the correctness of (1.14).

Representing the solution of (1.14) in the form (as in /1, 2/)

$$\varkappa = 2\varepsilon(1-\nu+\nu\delta_{i3})\mu^{-1}\sqrt{\rho^2(l)-m_1^2}\,Q_i(l,m_1,\varepsilon)\gamma^i(m_1,l)+o(\varepsilon^2) \tag{1.15}$$

we obtain for the stress intensity coefficients (SIC) of the normal, transverse, and longitudinal modes $K_1^{\pm}$, $K_2^{\pm}$, $K_3^{\pm}$

$$\begin{aligned} &K_1^{\pm}/K^0 = Q_2^{\pm}+o(\varepsilon), \quad K_2^{\pm}/K^0 = \pm Q_1^{\pm}-\varepsilon(1-\nu)^{-1}\rho' Q_3^{\pm}+o(\varepsilon),\\ &K_3^{\pm}/K^0 = \pm Q_3^{\pm}+\varepsilon(1-\nu)\rho' Q_1^{\pm}+o(\varepsilon), \quad K_i^{\pm}=K_i(\pm\rho(l),l,\varepsilon),\\ &Q_i^{\pm}=Q_i(\pm\rho(l)\ l\ \varepsilon) \quad K^0=\sqrt{\pi\varepsilon\rho(l)} \end{aligned} \tag{1.16}$$

The appearance of terms with $\rho'(l)$ in (1.16) is caused by the fact that for cracks with varying width $(\rho'(l)\neq 0)$ the direction along the crack contour does not coincide with the direction of the vector $\gamma^3(\pm\rho(l), l)$.

Starting from the relationships (1.14) and (1.16) and taking into account the geometrical meaning of the quantities $\hat{A}_i$, the dependence of the displacement jumps and the SIC on the "local" geometry parameters and loads can be described qualitatively; a change in the load and width of the crack during motion along the crack results in interaction of the longitudinal and transverse modes; rotation of the crack in its plane results in redistribution of the displacement jumps and stress intensity coefficients over the crack section (separately for each mode); flexure of the surface results in interaction between the normal and transverse modes; and twisting of the surface results in interaction between the longitudinal and transverse modes.

We present asymptotic formulas for the displacement jumps and SIC for certain typical load cases. According to (1.15) and (1.16), for this it is sufficient to give an expression for $Q_i$.

*A crack subjected to internal pressure p.* In this case $\mathbf{p}(m_1,l)=p\mathbf{n}(m_1,l)$ and we obtain from (1.20) and (1.21)

$$Q_1 = {}^1/_4(1-4\nu)A_1^*, \quad Q_2 = p+{}^1/_4 A_2^*, \quad Q_3={}^1/_2 A_3^*, \quad A_i^*=\varepsilon p m_1 A_i \tag{1.17}$$

Therefore (unlike the crack of plane planform), not only normal but also shear jumps in the displacement occur under the effect of internal pressure, where the magnitude and even the sign of the transverse displacement jumps depend substantially on Poisson's ratio $\nu$ because of the presence of the factor $(1-4\nu)$ in (1.17).

*A crack with unloaded edges in a stress field caused by certain loads at infinity.* In this case the state of stress and strain of an elastic medium is represented in the form of the sum of two elastic solutions: the state of stress and strain when there is no crack (with the stress tensor $\sigma^\infty(\mathbf{x})$) and a perturbation caused by application of the forces $\mathbf{p}=\sigma^\infty\mathbf{n}$ to the crack edges. Taking into account that for $\mathbf{x}\in G_\varepsilon$

$$\begin{aligned} &\sigma^\infty(\mathbf{x}) = \sigma^\infty(\mathbf{R}(l)+\varepsilon m_1\alpha^1(l)) = \sigma^\infty(\mathbf{R}(l))+\varepsilon m_1\sigma^\infty_{,1}(\mathbf{R}(l))+o(\varepsilon)\\ &\mathbf{p}(\mathbf{x}) = \sigma^\infty(\mathbf{R}(l))\alpha^2(l)+\varepsilon m_1\sigma^\infty_{,1}(\mathbf{R}(l))\alpha^2(l)-\varepsilon m_1 A_3\sigma^\infty(\mathbf{R}(l))\alpha^3(l)+o(\varepsilon) \end{aligned}$$

(where, the derivatives are taken in the local Cartesian coordinate system defined by the directions $\alpha^i(l)$), we obtain from (1.16) and (1.17)

$$\begin{aligned} &Q_i(m_1,l,\varepsilon) = \sigma^\infty_{2i}(\mathbf{R}(l))+{}^1/_2\varepsilon m_1 Q_i^*(l)\\ &Q_1^*(l) = \sigma^\infty_{12,1}-\nu(1-\nu)^{-1}\sigma^\infty_{23,3}+{}^1/_2(1-4\nu)A_1\sigma^\infty_{22}-A_3\sigma^\infty_{33}\\ &Q_2^*(l) = \sigma^\infty_{22,1}+{}^1/_2 A_2\sigma^\infty_{22}-{}^1/_2 A_2\sigma^\infty_{12}+{}^1/_2(1-4\nu)A_1\sigma^\infty_{12}-A_3[\sigma^\infty_{33}+\\ &\quad (2-\nu)(1-\nu)^{-1}\sigma^\infty_{23}]\\ &Q_3^*(l) = \sigma^\infty_{23,1}-\nu\sigma^\infty_{12,3}+{}^1/_2 A_2\sigma^\infty_{23}+A_3(2\sigma^\infty_{22}-\sigma^\infty_{33}) \end{aligned} \tag{1.18}$$

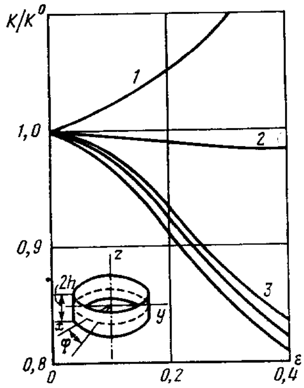
The values of the stress components and their derivatives in the last three equalities are calculated at the point $\mathbf{R}(l)$.

If the external stress field is homogeneous $(\sigma^\infty(x)=\sigma^\infty=\text{const})$, we have

$$\begin{aligned} &Q_1^* = {}^1/_2(1-\nu)^{-1}\{(1-3\nu)A_2\sigma_{12}^\infty+(1-3\nu+4\nu^2)A_1\sigma_{22}^\infty-\\ &\quad 2\nu A_1\sigma_{33}^\infty+2A_3[\nu\sigma_{13}^\infty-(1-\nu)\sigma_{33}^\infty]\}\\ &Q_2^* = {}^1/_2 A_2\sigma^\infty_{22}-{}^1/_2(1-4\nu)A_2\sigma^\infty_{12}-A_3[\sigma^\infty_{33}+(2-\nu)(1-\nu)^{-1}\sigma^\infty_{23}]\\ &Q_3^* = {}^1/_2(1+2\nu)A_2\sigma^\infty_{23}-\nu A_1\sigma^\infty_{13}+A_3[\nu\sigma^\infty_{11}+(2-\nu)\sigma^\infty_{22}-\sigma^\infty_{33}] \end{aligned} \tag{1.19}$$

Formulas (1.18) and (1.19) show, in particular, that during torsion of the crack surfaces the external stress components $\sigma_{13}^{\infty}, \sigma_{11}^{\infty}, \sigma_{33}^{\infty}$ which do not perturb cracks of planar planform, start indeed to exert an influence on the displacement jumps and the SIC.

2. **The axisymmetric problem of a crack on a cylindrical surface.** To verify the asymptotic formulas obtained above and to refine the influence of the crack surface curvature on the state of stress and strain, the problem of an axisymmetric crack on a cylindrical surface subjected to axisymmetric loads was analysed in greater detail (see the figure). The crack surface is determined in the cylindrical $r, \varphi, z$ coordinate system by the relationships $|z| < h, \quad r = R,$ where




$k(l) = 1/R, \rho(l) = R, \varepsilon = h/R, A_1(l) = k(l), A_2 = A_3 = 0$

$m_1 = z\varepsilon^{-1}, \quad \alpha^1 = \mathbf{e}_z, \quad \alpha^2 = \mathbf{e}_r, \quad \alpha^3 = \mathbf{e}_\varphi$

the load components $p_1 = p_z, \ p_2 = p_r, \ p_3 = p_\varphi$ and the displacement jumps $\varkappa_1 = \varkappa_z, \varkappa_2 = \varkappa_r, \varkappa_3 = \varkappa_\varphi$ are considered to be dependent only on the z coordinate. In this case, by integrating with respect to $\varphi$ in (1.7) and letting $r \to R$ in (1.6), exact one-dimensional integrodifferential equations of the problem can be obtained. The fundamental, most awkward, part of the calculations in deducing the above-mentioned equations was realized by using an electronic computer with the algorithmic language REDUCE-3 which enables algebraic manipulations to be performed in symbolic form. Consequently, equations of the following form are obtained (integration with respect to $z'$ is between the limits $-h$ and $h$):

$$\int K_{33}(z - z')\varkappa_3'(z')\,dz' = 2\pi D\mu^{-1}p_3(z) \tag{2.1}$$

$$\int K_{ij}(z - z')\varkappa_j'(z')\,dz' = \frac{D}{2}\beta p_i(z) \quad (i, j = 1, 2)$$

$$K_{11}(z) = [(\zeta^{-1} - \zeta)E + \zeta K]\,\zeta_1, \quad K_{33}(z) = [(\zeta^{-1} + 16\zeta)E - 8\zeta K + 16\zeta^3(E - K)]\,\zeta_1^3$$

$$K_{22}(z) = \{[\zeta^{-1} + (11 - 16\nu)\zeta + (28 - 32\nu)\zeta^3]E + [(1 + 8\nu)\zeta - (16 - 24\nu)\zeta^3]K + 16(1 - \nu)\zeta^5(E - K)\}\zeta_1^3$$

$$K_{21}(z) = -K_{12}(z) = \{[4 - 8\nu + (14 - 16\nu)\zeta^2]E - [1 - 4\nu + (10 - 12\nu)\zeta^2]K + 8(1 - \nu)\zeta^4(E - K)\}\,\zeta_1^3$$

$$D = 2R, \quad \zeta = z/D, \quad \zeta_1 = 1/\sqrt{1 + \zeta^2}$$

($K = K(\zeta_1)$, $E = E(\zeta_1)$ are complete elliptic integrals of the first and second kinds).

Comparing (2.1) and the asymptotic equations obtained from (1.14) for this case, we see that they are in complete agreement. Moreover, as might have been expected from symmetry considerations, problems for the normally-transverse and longitudinal loads are separated.

For the special case of an annular crack under axisymmetric loads, (2.1) enables us to refine the asymptotic form of the solution (1.15) and (1.16) for small $\varepsilon$ by the construction of appropriate power-law expansions (in $\varepsilon$) with any number of terms. To do this it is sufficient to change to the variable $m_1$ in (2.1), to expand the left and right sides in $\varepsilon$, to equate terms of like order and to solve, successively, the integrodifferential equations obtained.

Asymptotic displacement jumps and SIC were discussed for narrow cracks for a number of kinds of loads having the form of expansions in even or odd powers of $\varepsilon$ with coefficients dependent on $\lambda = \ln(16/\varepsilon), \nu$ and $M = m_1/R$ by using an electronic computer for the method described (from three to five terms of the asymptotic form are obtained in explicit form). Eqs.(2.1) were also solved numerically by the method of mechanical quadratures. When comparing the results obtained numerically and computed by the asymptotic formulas, the discrepancies in the range $0 \leqslant \varepsilon \leqslant 0{,}3$ were not more than 1-2%, which indicates the high accuracy of the asymptotic formulas. Good agreement is also observed between the results obtained and those presented in /4/ (the discrepancy is not more than 2-3%, which is within the limits of accuracy of the computations).

We will describe the SIC behaviour (determined as a result of calculations) for certain kinds of loads.

In the case of a constant longitudinal load $p_\varphi(z) = p_\varphi = \text{const}$, the dimensionless SIC $K_3/K_3^0\ (K_3^0 = p_\varphi\sqrt{\pi h})$ increases considerably as $\varepsilon$ increases (Figure, curve *1*). Under loading by a constant internal pressure $p_r(z) = p_r = \text{const}\ (p_z(z) = 0)$, the coefficient $K_1/K_1^0\ (K_1^0 = p_r\sqrt{\pi h})$ conversely decrease as $\varepsilon$ increases, where the greater the $\nu$, the more rapid the decrease (see the figure, curve *3*; the upper, middle, and lower curves correspond to $\nu = 0{,}1,\ 0{,}3$ and $0{,}5$). Here $K_2/K_1^0$ is much less than unity, increases in absolute value as $\varepsilon$ increases and depends very much on $\nu$ (which is in agreement with (1.17)).

In the case of a constant load along $z$, by solving (2.1) for $\varkappa_r, \varkappa_z$, for given $p_z(z) = p_z = \text{const},\ p_r(z) = 0$, we obtain a non-zero expansion $\varkappa_r$ which is an odd function in $z$ and therefore negative in a certain domain; such a solution has physical meaning only if the crack was first exposed or loaded by an additional exposing action (otherwise the interaction of the superposing crack edges must be taken into account). It is interesting to note that for the solution under consideration $K_2/K_2^0\ (K_2^0 = p_z\sqrt{\pi h})$ is practically independent of $\varepsilon$ and $\nu$ (the figure, curve *2*).

We construct the approximate solution for the case of a constant load in z taking the superposition of the crack edges into account assuming no friction between them. The problem here becomes the following: for $|z| \leqslant h$ it is required to find functions $\varkappa_r(z),\ \varkappa_z(z),\ p_r(z)$ satisfying the first two equations in (2.1) (with $p_z(z) = p_z = \text{const}$) and the conditions

$$\varkappa_r(z) \geqslant 0,\ p_r(z) \geqslant 0,\ \varkappa_r(z)\, p_r(z) = 0 \tag{2.2}$$

(The normal force $p_r(z)$ occurs because of superposition of the edges). For small $\varepsilon$ we seek the approximate solution in the form

$$\begin{aligned} \varkappa_z(m_1,\ \varepsilon) &= \varepsilon\,[\varkappa_z^0(m_1) + \varepsilon^2\varkappa_z^2(m_1) + o(\varepsilon^2)] \\ p_r(m_1,\ \varepsilon) &= \varepsilon p_r^1(m_1) + o(\varepsilon) \end{aligned} \tag{2.3}$$

The location of the superposition zone depends on the signs of $p_z$ and $1-4\nu$, furthermore, to be specific we consider $p_z > 0$ and $\nu > 0{,}25$ (the approximate solution is constructed analogously for the remaining cases). Eqs. (1.14), (2.2), (2.3) yield

$$\begin{aligned} &\varkappa_z^0 = 2(1-\nu)\,\mu^{-1} p_z^0 \sqrt{R^2 - m_1^2} \\ &2P(\varkappa_r^1)_{,\,m_1 m_1} = \beta p_r^1 - \beta p_z(1-4\nu)\, m_1/(2R) \\ &\varkappa_r^1 \geqslant 0,\quad p_r^1 \geqslant 0,\quad \varkappa_r^1 p_r^1 = 0 \end{aligned} \tag{2.4}$$

The second and third relationships in (2.4) determine the problem of a plane rectilinear crack with superposition of the edges whose solution, as is known, is unique. It is not complicated to seek this solution by starting from the assumption that the superposition domain is the segment $[-R, a]$. Then $\varkappa_r^1(m_1) = 0$ for $m_1 \in [-R, a]$, $p_r^1(m_1) = 0$ for $m_1 \in [a, R]$ and from the second Eq. (2.4) examined in $[a, R]$, we find $\varkappa_r^1$ on $[a, R]$, and then $-p_r^1$ on $[-R, a]$ from this same equation. It remains to seek an *a* such that the two equations (2.4) are satisfied. Such a value turns out to be $a = -R/3$, hence

$$\begin{aligned} \varkappa_r^1 &= \begin{cases} -(1-\nu)\,\mu^{-1}(1-4\nu)\, R^{-1}(m_1 + R/3)^{3/2}(R - m_1)^{1/2}, & -R/3 \leqslant m_1 \leqslant R \\ 0, \quad -R \leqslant m_1 \leqslant -R/3 \end{cases} \\ p_r^1 &= \begin{cases} 0, \quad -R/3 \leqslant m_1 \leqslant R \\ -{}^1/_2 p_z(1-4\nu)\, R^{-1}(2R/3 - m_1)\sqrt{(R/3 + m_1)/(m_1 - R)}, & -R \leqslant m_1 \leqslant -R/3 \end{cases} \end{aligned} \tag{2.5}$$

Using (2.5) and (2.1) we obtain

$$\varkappa_z^2 = {}^1/_{16}(1-\nu)\,\mu^{-1} p_z \sqrt{R^2 - m_1^2}\,\{2M^2 - 6\lambda + 15 + 4(1-4\nu)^2 u\}$$

$$u = \begin{cases} \dfrac{4}{27}\left[\arccos\ \dfrac{3M-1}{2} - \arccos M\right] + \dfrac{2 - M - 3M^2}{9}\times \\ \times\sqrt{(1-M)\left(M + \dfrac{1}{3}\right)}, \quad -\dfrac{1}{3} \leqslant M \leqslant 1 \\ \dfrac{4}{27}(\pi - \arccos M), \quad -1 \leqslant M \leqslant -\dfrac{1}{3} \end{cases}$$

$$\frac{K_1}{K_2^0} = \begin{cases} -(1-4\nu)\sqrt{2/27}\,\varepsilon + o(\varepsilon), & z = h \\ 0, \quad z = -h \end{cases}$$

$$\frac{K_2^{\pm}}{K_2^0} = \pm\left\{1 + \varepsilon^2\left[-\frac{3}{16}\lambda + \frac{17}{32} \mp \frac{(1-4\nu)^2}{54}\right] + o(\varepsilon^2)\right.$$

The external load in the problem is independent of *z*, consequently, the specific increment of the total potential elastic energy of the medium $\delta W/\delta S$ as the crack width $2h$ increases is independent of whether it would be achieved as a result of advancement of some contour ($z = h$ or $z = -h$). Therefore, in conformity with the Irwin formula, the quantity $\delta W/\delta S$ is identical on the contours $z = \pm h$ and equals

$$\delta W/\delta S = (1-\nu)\,\pi\mu^{-1}(K_2^0)^2\,\{1+\varepsilon^2[-3\lambda/8+17/16+(1-4\nu)^2/27]+o(\varepsilon^2)\}$$

In conclusion, we note that the action of even a small additional external pressure $p = \varepsilon K_p p_z$ on a crack edge results in a change in the superposition domain. By reasoning analogous to that presented above we obtain that the superposition domain is defined by the inequalities $-1 \leqslant M \leqslant K$, where $K = -1/3 - 8K_p/[3(4\nu-1)]$ and for the exposure we have

$$\varkappa_r \approx \begin{cases} 1/2\,(1-\nu)\,\mu^{-1}(4\nu-1)\,p_z h\sqrt{1-M}\,(M-K)^{3/2}, & K \leqslant M \leqslant 1 \\ 0, & -1 \leqslant M \leqslant K \end{cases}$$

For $p > \varepsilon K_p^1 p_z$, $K_p^1 = \nu - 1/4$ the crack is completely open, and completely closed for $p < \varepsilon K_p^2 p_z$, $K_p^2 = -2(\nu - 1/4)$.

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Translated by M.D.F.

*PMM U.S.S.R.*,Vol.51,No.5,pp.681-683,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00


# GREEN'S FUNCTION FOR THE BENDING OF A PLATE ON AN ELASTIC HALF-SPACE*

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Improper Woinowski-Krieger integrals /1/ expressing the deflections of an infinite plate and the contact reactions of an elastic half-space subjected to a unit normal force are considered. Elementary formulas to calculate the quantities mentioned in the neighbourhood of the point of application of the load are obtained from the power series expansion with a logarithm by Watson's method. The results of calculations using these formulas are in good agreement with the results of a numerical integration of quadratures /2/. The analytical representations obtained for Green's functions are convenient for utilization as kernels of the integral equations when solving contact problems for the interaction of bodies, one of which is reinforced by a thin covering.

Under the action of a unit normal force at a point with coordinates $(x_1, y_1)$ on an infinite plate lying without friction and adhesion on an elastic half-space, the deflections $w$ and contact pressures $p$ at a point with coordinates $(x, y)$ are expressed by the integrals /1, 3/

$$w = l^2(2D)^{-1}w_0,\quad p = l^{-2}p_0 \tag{1}$$

$$w_0 = \frac{1}{\pi}\int_0^\infty \frac{J_0(\lambda\rho)}{\lambda^3+1}\,d\lambda,\qquad p_0 = \frac{1}{2\pi}\int_0^\infty \frac{J_0(\lambda\rho)\,\lambda}{\lambda^3+1}\,d\lambda$$

$$\rho = l^{-1}((x-x_1)^2+(y-y_1)^2)^{1/2},\quad D = Eh^3(12(1-\nu^2))^{-1}$$

$$l = (2DE_0^{-1}(1-\nu_0^2))^{1/3}$$

Here $E, \nu$ are the elastic modulus and Poisson's ratio of a plate of thickness $h$ while

**Prikl.Matem.Mekhan.*,51,5,866-867,1987